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<u>Tests back</u>

8. $(A+B)^2 - A^2 - B^2 = 0$ $\qquad AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$(A+B)(A+B) - A^2 - B^2 = 0$ $\qquad BA = ?$

$A^2 + AB + BA + B^2 - A^2 - B^2 = 0$

$AB + BA = 0 \implies AB = -BA = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$

<u>Recall</u>

$V \subseteq \mathbb{R}^m$ subspace

A <u>basis</u> is a set $\{v_1, \dots, v_m\}$ such that

1) $v_1, \dots, v_m$ linearly independent

2) $\text{span}\{v_1, \dots, v_m\} = V$

<u>Ex:</u> Basis for $\mathbb{R}^3$: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

<u>Ex:</u> Let $v_1, \dots, v_m$ be a basis for $\mathbb{R}^n$

Then... $m = n$

<u>Pf.</u> Let $A = [\vec{v}_1, \dots, \vec{v}_m]$. Then $A\vec{x} = \vec{b}$ always has a solution. $\implies$ A has a pivot in every row

$\Rightarrow \vec{v}_1, \dots, \vec{v}_m$ lin. ind. $\Rightarrow$ A has a pivot in every column

$\Rightarrow m = n$ (must be square).

<u>Prop.</u>

A $m \times n$ matrix

1) if $n > m$, $A\vec{x} = \vec{0}$ has a nontrivial solution

$$\begin{array}{cc} & n \\ m & \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & & & 0 \end{array}\right] \end{array}$$

2) if $m > n$, $A\vec{x} = \vec{b}$ does not have a solution for some $\vec{b}$

$$\begin{array}{cc} & n \\ m & \left[\begin{array}{ccc|c} & & & \\ & & & \vec{b} \\ & & & \end{array}\right] \end{array}$$

<u>Thm</u> For $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^m$ the following are equivalent:

a) $\vec{v}_1, \dots, \vec{v}_m$ form a basis for $\mathbb{R}^m$

b) $A = [v_1 \cdots v_m]$ is invertible

c) $\vec{v}_1, \dots, \vec{v}_m$ span $\mathbb{R}^m$

d) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent

Thm Every nonzero subspace $V \subseteq \mathbb{R}^m$ has a basis

Ex: Find a basis for

$$\text{span}\left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \overset{v_5}{\begin{bmatrix} 4 \\ 6 \\ 3 \\ 7 \end{bmatrix}} \right\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$v_3 = v_2 - v_1$

Basis: $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Ex: Find bases for Col(A), Nul(A) $\quad A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}$

Soln. $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$

Col(A) Basis: $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Nul(A): solving $A\vec{x} = \vec{0}$

$$\begin{aligned} x_1 - x_3 + x_5 &= 0 \\ x_2 + x_3 + 2x_5 &= 0 \\ x_4 + x_5 &= 0 \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - x_5 \\ -x_3 - 2x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{Nul}(A) \text{ Basis: } \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Nul(A) Method: Solve $A\vec{x} = \vec{0}$, writing everything in terms of free variables, use the spanning set coming from that

Col(A) Method: Find RREF(A) and choose cols of A corresponding to pivots

Thm (Dimension) Let $V \subseteq \mathbb{R}^m$ be a subspace

Let $\vec{a}_1, \dots, \vec{a}_k \in V$ be linearly independent, and let $\vec{b}_1, \dots, \vec{b}_\ell$ span V. Then $k \leq \ell$

Pf. Let $A = [\vec{a}_1 \cdots \vec{a}_k]$, $B = [\vec{b}_1 \cdots \vec{b}_\ell]$

For each i, $\vec{a}_i = B\vec{c}_i$ for some c_i, $C = [c_1 \cdots c_k]$,

i.e. $BC = A$. C is $\ell \times k$, so if $k > \ell$, C has more cols than rows so some nonzero $\vec{v}$ has $C\vec{v} = \vec{0}$

$A\vec{v} = BC\vec{v} = \vec{0}$, contradicting independent columns of A. $\square$