2/12/25

Recap: A $m \times n$

$Col(A) =$ span of columns of A

$Nul(A) = \{ \vec{v} \mid A\vec{v} = \vec{0} \}$

Rmk: can always write column spaces as null spaces and vice versa

Ex: $A = \begin{bmatrix} 2 & 3 & -8 & -5 \\ -1 & 2 & -3 & -8 \end{bmatrix}$ Find B w/ $Nul(A) = Col(B)$

find $Nul(A)$:

$$\left[\begin{array}{cccc|c} 2 & 3 & -8 & -5 & 0 \\ -1 & 2 & -3 & -8 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} -1 & 2 & -3 & -8 & 0 \\ 2 & 3 & -8 & -5 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + 2R_1} \left[\begin{array}{cccc|c} -1 & 2 & -3 & -8 & 0 \\ 0 & 7 & -14 & -21 & 0 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cccc|c} -1 & 2 & -3 & -8 & 0 \\ 0 & 1 & -2 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cccc|c} -1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & -3 & 0 \end{array}\right]$$

$x_1 = x_3 - 2x_4$

$x_2 = 2x_3 + 3x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ 2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(going other way now)

<u>Ex:</u> $B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, Find C w/ $\mathrm{Nul}(C) = \mathrm{Col}(B)$

$$\mathrm{col}(B) = \{\vec{b} \mid B\vec{x} = \vec{b} \text{ has a solution}\}$$

$$\left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 2 & 3 & b_2 \\ 1 & 0 & b_3 \\ 0 & 1 & b_4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 0 & b_2 - 2b_1 - 5b_4 \\ 0 & 0 & b_3 - b_1 - b_4 \\ 0 & 1 & b_4 \end{array}\right]$$

$$b_2 - 2b_1 - 5b_4 = 0$$
$$b_3 - b_1 - b_4 = 0$$

$$C = \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} -2 & 1 & 0 & -5 \\ -1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

Bases

<u>Defn:</u> Let $V \subseteq \mathbb{R}^n$ be a subspace.

A <u>basis</u> of V is a set of vectors $\vec{v}_1, \dots, \vec{v}_m$ such that:

1) $V = \mathrm{span}\{\vec{v}_1, \dots, \vec{v}_m\}$
2) $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent

(Review for Exam 1)

Ex: $A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix}$ Find a matrix C w/ $CA = REF(A)$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix}$$

L_1 L_2 L_3

$$CA = L_3 L_2 L_1 A = REF(A)$$

$$C = L_3 L_2 L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

High-level

- (chapter 1) systems of linear equations
  - row reduction, pivots, free variables
- (first half of ch. 2) vectors, linear combinations of vectors, span, dependence/independence
- (second half of ch. 3) properties of matrices
  - products, sums, transpose, elementary matrices

back to start: $A\vec{x} = \vec{b}$