2/10/25

Ex: Do $\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\-5\\4\end{bmatrix}, \begin{bmatrix}4\\-3\\9\end{bmatrix}$ span $\mathbb{R}^3$?

They will span $\mathbb{R}^3$ $\iff$ $\begin{bmatrix}1 & 3 & 4\\-2 & -5 & -3\\1 & 4 & 9\end{bmatrix}$ is invertible

$\iff$ there is a pivot in every col

$$\xrightarrow[R_3-R_1]{R_2+2R_1} \begin{bmatrix}1 & 3 & 4\\0 & 1 & 5\\0 & 1 & 5\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1 & 3 & 4\\0 & 1 & 5\\0 & 0 & 0\end{bmatrix}$$

<u>No</u>

free var, not a pivot in every column $\Rightarrow$ do not span $\mathbb{R}^3$

<u>Prop</u> A $n \times n$. A invertible $\iff$ A^T invertible

$$(A^T)^{-1} = (A^{-1})^T$$

does this exist? i.e. A^T invertible?

we know this exists

<u>Recall</u>: $(AB)^T = B^T A^T$

<u>Pf.</u> Let A be invertible.

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

Similary, if A^T is invertible, then $(A^T)^T = A$ is too.

$\square$

Subspaces

$$A\vec{x} = \vec{b}$$

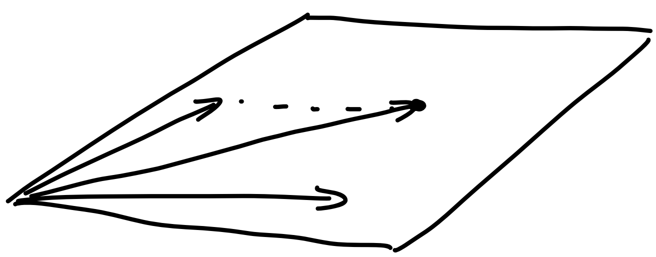
Recall: $A\vec{x} = \vec{b}$ is homogeneous if $\vec{b} = \vec{0}$

Thm) Every linear combination of solutions to $A\vec{x} = 0$ is again a solution

Pf. Let $\vec{v}_1, \ldots, \vec{v}_k$ be solutions to $A\vec{x} = 0$

$$A(c_1\vec{v}_1 + \ldots + c_k\vec{v}_k) = c_1 \underbrace{A\vec{v}_1}_{=0} + \ldots + c_k A\vec{v}_k$$

$$= \vec{0} + \ldots + \vec{0} = 0$$



visual example: linear combos of plane of solutions exist also in the plane

Defn: A subspace of $\mathbb{R}^n$ is a subset $V \subseteq \mathbb{R}^n$ satisfying:

1. $0 \in \mathbb{R}^n$ (non-empty) ← just checking not empty — $c\vec{u} \neq 0$ if no $\vec{u}$ exists!
2. If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$ (closed under +)
3. If $\vec{u} \in V$, $c \in \mathbb{R}$, then $c\vec{u} \in V$ (closed under scalar mult.)

Ex:

- $\mathbb{R}^n$ is a subspace of itself
- $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$
- $W = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$

↳ ex: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in W? no

$\begin{bmatrix} -3 \\ -6 \end{bmatrix}$ in W? yes!

Check: 1. $\vec{0} \in W$ (take $x=0$)

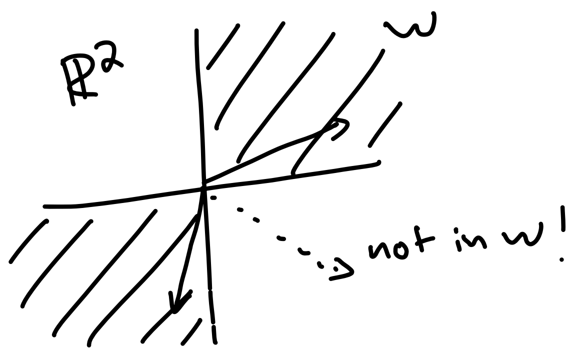
2. $\begin{bmatrix} x \\ 2x \end{bmatrix}, \begin{bmatrix} y \\ 2y \end{bmatrix} \in W$

$\Rightarrow \begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} y \\ 2y \end{bmatrix} = \begin{bmatrix} x+y \\ 2(x+y) \end{bmatrix} \in W$

3. $\begin{bmatrix} x \\ 2x \end{bmatrix} \in W,\ c \in \mathbb{R} \Rightarrow c\begin{bmatrix} x \\ 2x \end{bmatrix}$

$= \begin{bmatrix} cx \\ 2cx \end{bmatrix} \in W$

- $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, xy \geq 0 \right\}$



$\begin{bmatrix} -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix} \in W$

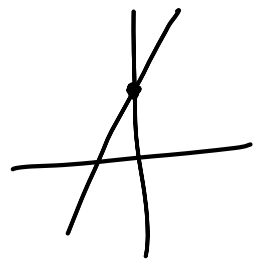
$\begin{bmatrix} -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \notin W$

W is not a subspace

Ex:

- $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, y = 2x+3 \right\}$ is not a subspace

$0 \notin W$



Thm: For $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, $\text{span}\{v_1, \dots, v_k\}$ is a subspace

Def^n: Let A be $m \times n$.

- The column space of A, $\text{Col}(A)$, is the span of the columns

- The null space of A, $\text{Nul}(A)$, is the set of vectors $\vec{v}$ w/ $A\vec{v} = \vec{0}$.

Prop: $\text{Col}(A)$ and $\text{Nul}(A)$ are subspaces