1/31/25

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad \ddots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_2 \end{cases}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

<u>Rmk:</u> if $A\vec{x} = \vec{b}$ has a solution

$\iff$

$\vec{b}$ is a linear combo of the cols of A

$\iff$

$\vec{b}$ is in the span of the cols of A

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \vec{x} = \vec{v}_1 x_1 + \vec{v}_2 x_2 + \cdots + \vec{v}_n x_n$$

Rmk: if $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ then $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent $\iff A\vec{x} = \vec{0}$ has a nontrivial solution

Ex: Is $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ in the span of $\left\{ \underset{v_1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \right\}$

i.e. linear combo of set $= \vec{u}$?

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{u}$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \end{array}\right] \xrightarrow{\frac{R_3}{3} \to R_3} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

no pivot in last col

Yes! exists a solution (linear combo)

Thm Let A, B be $m \times n$ matrices. If $A\vec{v} = B\vec{v}$ for every $\vec{v}$, then $A = B$

Motivation:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

generalization of $\hat{i}, \hat{j}, \hat{k}$

$$Ae_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$Ae_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$Ae_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Pf) Let e_i be the vector w/ i^{th} component $=1$ and all other components $=0$. Then for any matrix C, $C\vec{e}_i$ is the i^{th} column of C. Thus if $A\vec{e}_i = B\vec{e}_i$ for all i, then A and B have the same colums, so are the same.

— — —

Ex: Find an $n \times n$ matrix with the property

$$A\vec{v} = \vec{v} \text{ for all } \vec{v}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \\ 0 & & 1\ldots \end{bmatrix}\vec{v}$$

Def^n: The $n \times n$ <u>identity matrix</u> is the matrix whose columns are the n standard cooordinate vectors in $\mathbb{R}^n$, meaning e_i

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 \cdots & 0 \\ 0 & \vdots & 0 & 1 & 0 \cdots \\ & & \vdots & & \ddots \end{bmatrix}$$

Matrix Multiplication

Def^n: If A is $m \times n$ and B is $n \times p$, then AB is the $m \times p$ $[A v_1 \cdots A v_p]$

where $B = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_p \\ | & & | \end{bmatrix}$

Alt:

AB is the matrix whose ij entry is the ith row of A dotted with the j^{th} column of B.

Rmk:

#cols of A must $=$ # rows of B for AB to make sense.

Ex: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \left[\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} -2 & 3 & 2 \\ 4 & 6 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & -1 & 2 & 5 \\ 3 & 0 & 1 & 1 \\ -2 & 3 & 5 & -3 \end{bmatrix}$

$$AB = \begin{bmatrix} -3 & 8 & 9 & -13 \\ 38 & -10 & 4 & 32 \end{bmatrix}$$

Ex: $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$CD = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad DC = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

<u>Warning!</u> AB can be different from BA even when both are defined

- $AB = AC$ <u>does not</u> imply $B = C$

- It's possible for $AB = 0$ even if A and $B \neq 0$

Ex: $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

<u>Fact</u>: still have $A(BC) = (AB)C$

<u>Defn</u>: A square matrix is a $n \times n$ matrix

And for a square matrix, we define

$A^2 = AA, \quad A^3 = AAA, \text{ etc.}$

<u>Weekend Funsies:</u>

$A = \begin{bmatrix} 2 & 7 \\ 6 & 3 \end{bmatrix} \quad A^3 = ?$

$$A^3 = AAA = AA(A) = \begin{bmatrix} 2 & 7 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 6 & 3 \end{bmatrix} A = \begin{bmatrix} 4+42 & 14+21 \\ 12+18 & 42+9 \end{bmatrix} A$$

$$= \begin{bmatrix} 46 & 35 \\ 30 & 51 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 92+210 & 322+105 \\ 60+306 & 210+153 \end{bmatrix} = \begin{bmatrix} 302 & 427 \\ 366 & 363 \end{bmatrix}$$