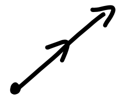
1/29/25

Quiz Recap: ① linearly dependent means exists nontrivial linear combination $= \vec{0}$

② <u>T or F</u>

1) F  2 vectors don't always span $\mathbb{R}^2$

2) T 

3) F 
- nonparallel $\neq$ independent (for $\mathbb{R}^n \mid n > 2$)
- don't span

4) F

5) True, need at least n vectors to span $\mathbb{R}^n$

<u>Big Picture</u>

① $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_2 \end{array}\right]$ is consistent if and only if $\begin{bmatrix} b_1 \\ \vdots \\ b_2 \end{bmatrix}$ is in the span of $\begin{bmatrix} a_{11} \\ \vdots \\ a_{1m} \end{bmatrix}, \ldots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$

(the columns)

② If it is consistent, then it has unique solution if and only if $\begin{bmatrix} a_{11} \\ \vdots \\ a_{1m} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$ are linearly independent

$\Downarrow$

think: $\vec{b}$ doesn't have to be $\vec{0}$ to do same process $\rightarrow$ (no free vars, pivots in each)

Matrix Algebra

Def: Let A be a $m \times n$ matrix w/ cols. $\vec{v}_1, \dots, \vec{v}_k$

The product of A w/ $\vec{x} \in \mathbb{R}^n$ is:

$$A\vec{x} = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$$

$$\begin{array}{c} \| \end{array}$$

$$m \left[\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underbrace{\hphantom{\begin{bmatrix} | & & | \end{bmatrix}}}_{n}$$

This is a vector in $\mathbb{R}^m$

Upshot: $A\vec{x} = \vec{b}$ means a particular linear combination of the <u>cols.</u> of A is equal to b.

Ex: $$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 4 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 8 \end{bmatrix} + 3\begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 32 \\ 50 \end{bmatrix}$$

Recall: dot product of $1 \times n$ and $n \times 1$ vector:

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \dots + a_n b_n$$

Alternate way to do $A\vec{x}$:

Dot the rows of A w/ $\vec{x}$

Ex: $\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4\times 1 + 5\times 2 + 6\times 3 \\ 7\times 1 + 8\times 2 + 9\times 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 50 \end{bmatrix}$

Properties of Matrix-vector product:

- $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\vec{u}) = cA\vec{u}$

where A $m\times n$ matrix, $\vec{u},\vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

Key Abstraction:

Solutions to

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \vdots \qquad \ddots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_n \end{cases}$$

are the same as solutions to

$A\vec{x} = \vec{b}$, where $A = [a_{ij}]$ $\begin{matrix} i = \{1,\dots,m\} \\ j = \{1,\dots,n\} \end{matrix}$

$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Ex: Solve $\begin{bmatrix} 2 & 0 & 2 \\ 4 & -1 & 6 \\ 1 & 3 & -5 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 4 & -1 & 6 & -5 \\ 1 & 3 & -5 & 15 \end{array}\right] \xrightarrow{\text{REF}} \cdots \longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

free

$$x_1 + x_3 = 0$$
$$x_2 - 2x_3 = 5$$
$$x_3 = x_3$$

$$\vec{x} = \begin{bmatrix} -x_3 \\ 2x_3 + 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$