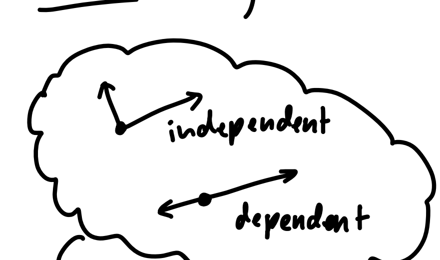
1/27/25

Recall: A system of equations $[A \mid \vec{b}]$ is homogenenous if $\vec{b} = \vec{0}$

Rmk: Homogeneous systems always have the trivial solution $x_1 = \dots = x_n = 0$, so homogeneous solutions are always consistent.

- It has a nontrivial solution $\Longleftrightarrow$ it has a free variable
- For a homogenous system, the last column is all 0's as we row reduce, so we can omit it.

Facts: a) Two vectors are linearly dependent $\Longleftrightarrow$ one is a scalar multiple of the other (they are parallel)



recall:
lin. dep. if
$c_1 v_1 + \dots + c_k v_k = 0$
(if not all trivially 0)

Pf. a) If $v_1 = c v_2$, then $v_1 - c v_2 = 0$ is a nontrivial relation ($c_1 = 1$, $c_2 = -c$).

If v_1, v_2 are linearly dependent, then if $c_1 v_1 + c_2 v_2 = 0$, and one of the c_i, say c_1, is not 0, then

$$c_1 v_1 = -c_2 v_2$$

$$v_1 = \frac{-c_2}{c_1} v_2$$

$\square$

<u>Facts contd.</u> b) Any set containing the zero vector is dependent

ex: $v_1, \dots, v_k, \vec{0}$ is dependent, since

$$0\vec{v}_1 + \dots + 0\vec{v}_k + 5 \cdot \vec{0} = \vec{0}$$

c) If a subset of $\{v_1, \dots, v_k\}$ is linearly dependent, so is $v_i, \dots, v_k$

ex: If $v_1, \dots, v_j$ are dep, i.e. $c_1 v_1 + \dots + c_j v_j = 0$ with some $c_i \neq 0$ for $i \leq j$, then $c_1 v_1 + \dots + c_j v_j + 0 v_{j+1} + \dots + 0 v_k = 0$

<u>Thm</u>: $\{\vec{v}_1, \dots, \vec{v}_k\}$ is lin dep $\Longleftrightarrow$ one of the vectors is in the span of the others,

<u>Rmk</u>: Any <u>such vector</u> can be removed without affecting the span.

<u>Pf</u>: If $v_k = c_1 v_1 + \dots + c_{k-1} v_{k-1}$, then

$$c_1 v_1 + \dots + c_{k-1} v_{k-1} - v_k = 0$$ is a non-trivial relation

If $c_1 v_1 + \dots + c_k v_k = 0$ is a nontrivial relation, then some c_i, say c_k, is nonzero. Then,

$$v_k = -\frac{1}{c_k}\left(c_1 v_1 + \dots + c_{k-1} v_{k-1}\right) \quad \square$$

True or False: If $\{v_1, \dots, v_k\}$ are linearly dependent, then every v_j is in the span of the others, e.g., v_k is in the span of $\{v_1, \dots, v_k\}$

False! counter ex:

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

$$-2v_1 + v_2 + 0v_3 = 0 \Rightarrow \text{dependent}$$

But, $v_3 \neq c_1 v_1 + c_2 v_2$

Thm: $\{v_1, \dots, v_k\}$ is linearly independent $\iff$ for each j, $\vec{v}_j$ is not in the span of $\{\vec{v}_1, \dots, \vec{v}_{j-1}, \vec{v}_{j+1}, \dots, \vec{v}_k\}$.

(follows from prev one)

Ex: Are $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ linearly independent?

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \end{array} \xrightarrow[R_4 - 2R_1]{R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow[R_4 + 3R_2]{R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Yes!

(no free vars)