11/24/25

(from end of last class)

Ex: what is $\text{span}\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\-3\\4\end{bmatrix}\right\}$?

$$\text{span} = \left\{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \;\middle|\; c_1\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2\begin{bmatrix}-2\\3\\0\end{bmatrix} + c_3\begin{bmatrix}4\\-3\\4\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \text{ has a soln.}\right\}$$

$$\begin{aligned} c_1 - 2c_2 + 4c_3 &= b_1 \\ 3c_2 - 3c_3 &= b_2 \\ 2c_1 + 4c_3 &= b_3 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 4 & b_1 \\ 0 & 3 & -3 & b_2 \\ 2 & 0 & 4 & b_3 \end{array}\right]$$

$$\longrightarrow \ldots \text{REF} \ldots \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 4 & b_1 \\ 0 & 3 & -3 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - \frac{4}{3}b_2 \end{array}\right]$$

system is consistent $\iff b_3 - 2b_1 - \frac{4}{3}b_2 = 0$

$$\text{span}\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\-3\\4\end{bmatrix}\right\} = \left\{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \;\middle|\; b_3 - 2b_1 - \frac{4}{3}b_2 = 0\right\}$$

Q: is $\begin{bmatrix}7\\1\\2\end{bmatrix}$ in the span?

A: $2 - 2(7) - \frac{4}{3}(1) = -12 - \frac{4}{3} \neq 0$ no.

Ex: what is the span of

$$\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\2\\0\end{bmatrix} ? \implies \text{all } \mathbb{R}^3$$

$$\begin{bmatrix}1&2&-2\\2&0&2\\1&1&0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c}1&0&0&b_1\\0&1&0&b_2\\0&0&1&b_3\end{array}\right]$$

always consistent

Ex: $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$, A is RREF of $\vec{v}_1, \vec{v}_2, \vec{v}_3$

what does $\operatorname{span}\{v_1, v_2, v_3\}$ look like if:

- A has one pivot? line $\longleftarrow \left[\begin{array}{ccc|c}1&0&0&\\0&0&0&*\\0&0&0&*\end{array}\right]$ 2 equations, intersecting planes
- A has two pivots? plane $\longleftarrow \left[\begin{array}{ccc|c}1&0&0&\\0&1&0&\\0&0&0&*\end{array}\right]$
- A has three pivots? $\mathbb{R}^3$

linear polynomial (must $= 0$, makes plane)

<u>Def^n:</u> A set $\{v_1, \dots, v_k\}$ is <u>linearly independent</u> if $x_1\vec{v}_1 + \dots + x_k\vec{v}_k = \vec{0}$ has only the trivial solution $x_1 = \dots = x_k = 0$

A set is <u>linearly dependent</u> otherwise, i.e. there are $x_1, \dots, x_k$ not all 0 s.t. $x_1\vec{v}_1 + \dots + x_k\vec{v}_k = \vec{0}$

Ex: ① Are $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ linearly dependent?

Yes, $2\begin{bmatrix}1\\2\end{bmatrix} - 1\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

② Are $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ linearly dependent?

$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\left[\begin{array}{cc|c}1 & 2 & 0\\ 2 & 3 & 0\end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c}\boxed{1} & 2 & 0\\ 0 & \boxed{-1} & 0\end{array}\right]$

$\Rightarrow$ no free variables $\Rightarrow$ unique solution $\Rightarrow$ linearly independent

③ If $\vec{u}, \vec{v}$ are linearly independent, show $\vec{v}$, $\vec{u}+\vec{v}$ are also linearly independent

Pf: suppose $\vec{u}, \vec{v}$ linearly independent. If $c_1(\vec{u}+\vec{v}) + c_2\vec{v} = \vec{0}$

$\Rightarrow c_1\vec{u} + (c_1+c_2)\vec{v} = 0 \Rightarrow c_1 = 0,\ c_1 + c_2 = 0$

$\Rightarrow c_1 = 0,\ c_2 = 0$ $\square$

④ Are there 3 linearly independent vectors in $\mathbb{R}^2$?

4 linearly independent vectors in $\mathbb{R}^3$?

No! – If you have more cols than rows, you must have a free variable $\Rightarrow$ linearly dependent

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$ linearly independent?

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & 4 & 0 \end{array}\right] \longrightarrow \ldots \text{REF} \ldots \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable

No! Infinitely many solutions $\Longrightarrow$ linearly dependent

Defn: A system of linear equations

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

is homogenous if $b_i = 0$ for all i