1/22/25

- Quiz @ start of class (free variables = infinite solutions!)

Vectors

$\mathbb{R}^2 = \{(x,y)\}$

$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ the set of n-tuples of real numbers. We call elements of $\mathbb{R}^n$ <u>vectors</u>

Can think of vectors as $n \times 1$ matrices

n ← rows, 1 ← columns

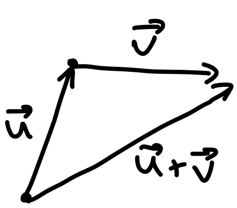
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$ — we sometimes write $\vec{v} = (v_1, v_2, \dots, v_n)$ to save space

Two vectors are equal $\iff$ they have the same coordinates,

i.e. $\vec{u} = \vec{v} \iff u_i = v_i$ for all i

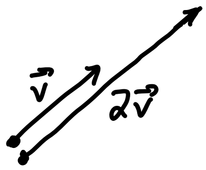
Vector operations

<u>Addition:</u> $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$



Scalar Multiplication: $c \in \mathbb{R}, \quad v \in \mathbb{R}^n$

$$c\vec{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_2 \end{bmatrix}$$



Special vector:

$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ the zero vector

for all $\vec{v} \in \mathbb{R}^n$, $\vec{v} + \vec{0} = \vec{v}$

Rmk:

- $c + \vec{v}$ doesn't make sense
- $\vec{u} \cdot \vec{v}$ doesn't make sense (yet!)
- write $-\vec{v} = (-1)\vec{v}$
- Usual arithmetic laws apply, e.g. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

  $a(b\vec{v}) = (ab)\vec{v}$

Defn: Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. A linear combination of $v_1, \dots, v_k$ is $c_1 v_1 + \dots + c_k v_k$ for some $c_1, \dots, c_k \in \mathbb{R}$

The $c_1, \dots, c_k$ are called the weights or coefficients of the $\vec{v}_i$.

Ex: Is $\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$?

$$\begin{bmatrix} c_1 - c_2 \\ 2c_1 - 2c_2 \\ 6c_1 - c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} \rightsquigarrow \begin{cases} c_1 - c_2 = 8 \\ 2c_1 - 2c_2 = 16 \\ 6c_1 - c_2 = 3 \end{cases}$$

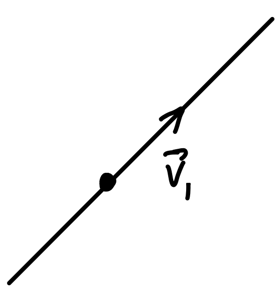
$$\longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right] \xrightarrow{\text{(row reduce)}} \cdots \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -9 \end{array}\right] \longrightarrow \begin{matrix} \underline{\text{Yes}} \\ c_1 = -1 \\ c_2 = -9 \end{matrix}$$

i.e. $\underline{-1} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + \underline{-9} \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$

<u>Defn</u>: For $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, their <u>span</u>, written as $\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, is the set of all linear combos of $\vec{v}_1, \ldots, \vec{v}_k$, i.e. $\{c_1\vec{v}_1 + \ldots + c_k\vec{v}_k \mid c_i \in \mathbb{R}\}$

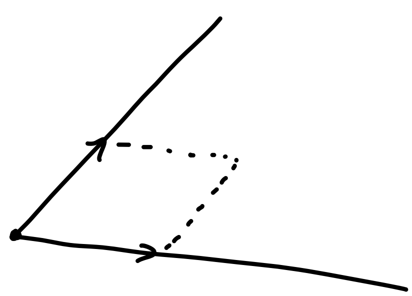
<u>Picture</u> $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$ "random" vectors

1) Span $\vec{v}_1$



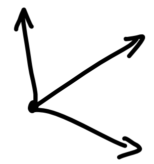
line through origin

2) Span $\{\vec{v}_1, \vec{v}_2\}$



plane through origin

3) span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ all of $\mathbb{R}^3$



<u>Ex:</u> What is span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 4 \end{bmatrix} \right\}$?

start: $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$