1/17/25

Recap: Row Echelon Form (REF): get 0's in lower left

Reduced Row Echelon Form (RREF): pivot is only nonzero entry in each column

To go from REF to RREF:

- scale each row so pivots are 1
- for each pivot, add multiples of that row to the rows above to get 0's above the pivot
- start w/ bottom row

Ex: Solve $\begin{cases} 2x+10y=1 \\ 3x+15y=2 \end{cases}$ $\left[\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 3 & 15 & 2 \end{array}\right]$

$\xrightarrow{-3R_1} \left[\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{array}\right]$ no solutions (says $0=\frac{7}{2}$)

system is inconsistent

Defn: A pivot position is an entry that becomes a pivot in a REF of the matrix

Defn: A pivot column is a column containing a pivot position

Ex: Find pivot pos & cols of

pivot columns

$\left[\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & 6 & 12 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$

$$\longrightarrow \quad \cdots \quad \longrightarrow \left[\begin{array}{ccc|c} \underline{1} & 2 & 3 & 6 \\ 0 & \underline{1} & 2 & 4 \\ 0 & 0 & \underline{1} & 3 \end{array}\right]$$

<u>Fact</u>: Every matrix is row equivalent to a unique RREF matrix

<u>Ex</u>: Solve $\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \qquad \begin{array}{l} x + 5z = 1 \\ y + 2z = -1 \end{array} \longrightarrow \begin{array}{l} x = 1 - 5z \\ y = -1 - 2z \\ z = \text{anything!} \end{array}$$

called a free variable

<u>Prop</u>: A system of linear equations is inconsistent if and only if the last column of the augmented matrix is a pivot column

<u>Proof</u>: put the system in RREF. If the last column is a pivot column, then some row is all 0's followed by a 1 in the last column. $\left[\begin{array}{ccc|c} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{array}\right] \to 0 = 1 \to$ inconsistent

If the last column does not contain a pivot, set variables for the non-pivot columns equal to 0. Then we can solve for the pivot vars in terms of the constants to get a solution

$$\left[\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Have RREF: $\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{array}\right] \end{array}$

Def^n: In a system of linear equations, a <u>free variable</u> is a variable corresponding to a column with no pivot

Upshot: Can solve for the pivot variables in terms of the free variables

To solve a linear system:

- make augmented matrix
- put in REF
- if last column is pivot column, no solutions
- else: - put in RREF
  - solve for pivot variables in terms of free variables

Rmk: We have infinitely many solutions if and only if the last column is not a pivot <u>and</u> there is a free variable (think parametric)

If a consistent system has finitely many solutions, have only 1 solution