1/15/25

<u>3 operations</u> (recap)

- swap rows
- multiply row by nonzero constant
- replace a row with sum of itself and any multiple of another row

can be used to solve any system of linear eqns

<u>Ex:</u> Solve $\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{augmented matrix}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$

$\xrightarrow{R2 - 2R1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R3 - 3R1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$

$\xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right] \xrightarrow{-\frac{1}{5}R2} \left[\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right]$

$\xrightarrow{R3 + 7R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right] \xrightarrow{\frac{1}{10}R3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$10z = 30$

$\xrightarrow{\substack{R1 - 3R3 \\ R2 - 2R3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R1 - 2R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$x = 1$
$y = -2$
$z = 3$

Defn: Two matrices are called *row equivalent* if you can get from one to the other using row operations.

Rmk: Any two row equivalent matrices have the same solution set

Ex: Solve $\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right] \xrightarrow[R3-4R1]{R2-3R1}$

$\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right] \longleftarrow 0 = 2$

no solutions!

<u>we now formalize how to solve systems of equations:</u>

Defn: A matrix is in *row echelon form* (REF) if:

1. All zero rows are at the bottom
2. The first nonzero entry of a row is to the right of the first nonzero entry of the row above
3. Below the first nonzero entry of a row, all other entries are zero

pivots

Ex: $\begin{bmatrix} \underline{3} & 7 & 9 & 8 \\ 0 & 0 & \underline{2} & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is REF $\qquad \begin{bmatrix} \underline{3} & 0 & 2 & 1 \\ 0 & 0 & \underline{1} & 2 \end{bmatrix}$ is REF $\qquad \begin{bmatrix} 0 & \underline{3} & 2 & 5 \\ \underline{1} & 2 & 3 & 4 \end{bmatrix}$ is <u>not</u> REF

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ is REF $\quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$ is not REF

Defn: A pivot is the first nonzero entry of a row of a matrix in REF

Defn: A matrix is in reduced row echelon form if it is in REF and:

4. Each pivot has value 1
5. Each pivot is the only nonzero entry in its column

To put a matrix in REF:

1) Find 1st nonzero column. Swap so that R1 has nonzero entry in that colum

2) Multiply R1 by const. so 1st nonzero entry of R1 is 1

3) Add multiples of R1 to other rows so that the leading 1 in R1 is the only nonzero entry in that column

4) Ignore R1 and repeat.

Ex: $\left[\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right]$

$R_1 \longleftrightarrow R_2$

$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 12 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$

$\frac{1}{2}R1$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$

Time Complexity: $O(n^3)$